Feedback Stabilization over Commutative Rings with no Right-/Left-Coprime Factorizations

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1 Introduction

Anantharam showed in [1] the existence of a model in which some stabilizable plants do not have its right-/left-coprime factorizations. In this paper, we give a condition of the nonexistence of the right-/left-coprime factorizations of stabilizable plants as a generalization of Anantharam's result. As examples of the models which satisfy the condition, we present two models; one is Anantharam's example and the other the discrete finite-time delay system which does not have the unit delay. We illustrate the construction of stabilizing controllers of stabilizable single-input single-output plants of such models. The method presented here is an application of the result of the necessary and sufficient condition of the stabilizability over commutative rings, which has recently been developed by Abe and the author[2].

2 Preliminaries

The reader is referred to Section 2 of [3] for the notations of commutative rings, matrices, and modules commonly used throughout the paper, for the formulation of the feedback stabilization problem, and for the related previous results.

3 Anantharam's Result and Its Generalization

3.1 Anantharam's Result

In [1], Anantharam considered the case where $\mathbb{Z}[\sqrt{5}i]$ ($\simeq \mathbb{Z}[x]/(x^2+5)$) is the set of the stable causal transfer functions, where $\mathbb{Z}$ is the ring of integers and i the imaginary unit; that is, $\mathcal{A} = \mathbb{Z}[\sqrt{5}i]$. The set of all possible transfer functions is given as the field of fractions of $\mathcal{A}$; that is, $\mathcal{F} = \mathbb{Q}(\sqrt{5}i)$. He considered the single-input single-output case and showed that the plant $p = (1+\sqrt{5}i)/2$ does not have its coprime factorization over $\mathcal{A}$ but is stabilizable. As a result, the question "*Is it always necessary that the plant and its stabilizing controller individually have coprime factorizations when the closed loop is stable?*" posed in [4] was negatively answered over general commutative rings.

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3.2 A Generalization

The result given by Anantharam[1] can be generalized over commutative rings.

Proposition 3.1 *If there exist $a, b, a', b' \in \mathcal{A}$ satisfying the following three statements, then there exists a causal stabilizable plant P which does not have right-/left-coprime factorizations:*

(i) The equality $ab = a'b'$ holds.

(ii) The ratio a/a' is a causal transfer function and does not have coprime factorization.

(iii) The pair (a, b) is coprime over $\mathcal{A}$. ∎

To prove the proposition above, we use the following proposition for the single-input single-output case:

Proposition 3.2 *Let $a, b, a', b' \in \mathcal{A}$. Suppose that $ab = a'b'$ holds. Suppose further that a/a' is a causal plant p. Then, even if the plant p does not have a coprime factorization, if (a, b) is coprime, then the plant p is stabilizable.*

Proof. We employ the notations in Definition 2.4 of [3]. Thus, $\mathcal{I} = \{\{1\}, \{2\}\}$, say $I_1 = \{1\}$ and $I_2 = \{2\}$. Then the generalized elementary factors of the plant are given as follows:

$$\begin{aligned} \Lambda_{pI_1} &= \{\lambda \in \mathcal{A} \,|\, \exists K \in \mathcal{A}^{(m+n)\times m} \; \lambda T = K \Delta_{I_1} T\} \\ &= \{\lambda \in \mathcal{A} \,|\, \lambda a' a^{-1} \in \mathcal{A}\}, \\ \Lambda_{pI_2} &= \{\lambda \in \mathcal{A} \,|\, \exists K \in \mathcal{A}^{(m+n)\times m} \; \lambda T = K \Delta_{I_2} T\} \\ &= \{\lambda \in \mathcal{A} \,|\, \lambda a a'^{-1} \in \mathcal{A}\}. \end{aligned} \tag{1}$$

It is obvious that $a \in \Lambda_{pI_1}$. On the other hand, b is a member of Λ_{pI_1} since the ratio aa'^{-1} in (1) can be rewritten as $b'b^{-1}$. Since (a, b) is coprime, we have $\Lambda_{pI_1} + \Lambda_{pI_2} = \mathcal{A}$. Hence by Theorem 2.1 in [3] the plant is stabilizable. □

Proof of Proposition 3.1. From the conditions (i) and (iii), and Proposition 3.2, the plant a/a' is stabilizable and does not have right-/left-coprime factorization by the condition (ii). □

We note that Proposition 3.1 gives a condition of the *nonexistence* of the doubly coprime factorization of the stabilizable plant. On the other hand, it should be noted that Sule in [5] has given a condition of the existence of the doubly coprime factorization of the stabilizable plant, which is expressed as follows:

Proposition 3.3 (Theorem 3 of [5]) *Let $\max \mathcal{A}$ be Noetherian and $\dim \max \mathcal{A} = 0$, where $\max \mathcal{A}$ denotes the set of all maximal ideals of $\mathcal{A}$. Then the plant is stabilizable if and only if it has a doubly coprime factorization.* ∎

In particular, if the plant is of the single-input single-output and if $\mathcal{A}$ is a unique factorization domain, then Raman and Liu in [6] gave the result that the plant is stabilizable if and only if it has a doubly coprime factorization.

We now give two examples which satisfy the condition of Proposition 3.1.

Example 3.1 In the example given in [1], that is, the case where $p = (1 + \sqrt{5}i)/2$, the numbers $1 + \sqrt{5}i$, $1 - \sqrt{5}i$, 2, 3 are corresponding to the variables a, b, a', b', respectively, in Proposition 3.1. ∎

We can make analogous examples. For example, let $\mathcal{A} = \mathbb{Z}[\sqrt{xy-1}i]$, where the integers x and y satisfy the following conditions: (i) $\gcd(x, y) = 1$ over $\mathbb{Z}$, (ii) $y > x \geq 2$, (iii) $xy - 1$ is not square. If the plant $p = (1 + \sqrt{xy-1}i)/x$, then the numbers $1+\sqrt{xy-1}i$, $1-\sqrt{xy-1}i$, x and y are corresponding to a, b, a', and b', respectively, in Proposition 3.1.

Example 3.2 Let us consider the discrete finite-time delay system. On some high-speed electronic circuits such as computer memory devices, they cannot often have nonzero small delays. We suppose here that the system cannot have the unit delay as a nonzero small delay. In this case, the set $\mathcal{A}$ becomes the set of polynomials generated by x^2 and x^3, that is, $\mathcal{A} = \mathbb{R}[x^2, x^3]$, where x denotes the unit delay operator. Then $\mathcal{A}$ is not a unique factorization domain but a Noetherian domain. The set $\mathcal{Z}$ used to define the causality is given as the set of polynomials in $\mathbb{R}[x^2, x^3]$ whose constant terms are zero; that is, $\mathcal{Z} = \{\alpha x^2 + \beta x^3 \,|\, \alpha, \beta \in \mathcal{A}\}$.

Let us suppose that $p = (1 - x^3)/(1 - x^2) \in \mathcal{P}$. Since $(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4)$, the plant can be also expressed as $p = (1 + x^2 + x^4)/(1 + x^3)$. Then $(1 - x^3)$, $(1 + x^3)$, $(1 - x^2)$, and $(1 + x^2 + x^4)$ are corresponding to a, b, a', b' in Proposition 3.2. Hence the plant does not have its coprime factorization but is stabilizable. ∎

4 Stabilizability and Construction of Stabilizing Controllers

In this section we present first the stabilizability for models of Examples 3.1 and 3.2, and then the construction of stabilizing controllers plants.

4.1 Stabilizability

The following two propositions give the stabilizability of all transfer functions in both cases of $\mathcal{A} = \mathbb{Z}[\sqrt{5}i]$ and $\mathcal{A} = \mathbb{R}[x^2, x^3]$.

Proposition 4.1 *Let $\mathcal{A} = \mathbb{Z}[\sqrt{5}i]$. Suppose that $\mathcal{Z} = \{0\}$, so that $\mathcal{P} = \mathcal{F}$. Then any transfer functions in $\mathcal{F}$ are stabilizable.*

Proof. In the case $p = 0$, the plant p is obviously stabilizable. Hence in the following we assume without loss of generality that $p \neq 0$. Let us consider a plant p is expressed as $p = (\alpha_1 + \alpha_2\sqrt{5}i)/\beta$, where $\alpha_1, \alpha_2, \beta \in \mathbb{Z}$. Let g denote $\gcd(\alpha_1^2 + 5\alpha_2^2, \beta)$ over $\mathbb{Z}$. Let α' be an integer such that $\alpha_1^2 + 5\alpha_2^2 = \alpha' g$. We note that α' does not have β as a factor.

To show the stabilizability, we here apply (iii) of Theorem 2.1 in [3]. Since the plant is of the single-input single-output, the set $\mathcal{I}$ defined in Definition 2.4 of [3] is equal to $\{\{1\}, \{2\}\}$, say $I_1 = \{1\}$ and $I_2 = \{2\}$. Then the generalized elementary factors Λ_{pI_1} and Λ_{pI_2} are given as follows:

$$\Lambda_{pI_1} = \{\lambda \in \mathcal{A} \,|\, \lambda d n^{-1} \in \mathcal{A}\}, \Lambda_{pI_2} = \{\lambda \in \mathcal{A} \,|\, \lambda n d^{-1} \in \mathcal{A}\}. \qquad (2)$$

Then one can check that $\alpha' \in \Lambda_{pI_1}$ and $\beta \in \Lambda_{pI_2}$. Since $\gcd(\alpha', \beta) = 1$ over $\mathbb{Z}$, we have $\Lambda_{pI_1} + \Lambda_{pI_2} = \mathcal{A}$. Hence any plant p is stabilizable by Theorem 2.1 in [3]. $\square$

Proposition 4.2 *In the case of $\mathcal{A} = \mathbb{R}[x^2, x^3]$, any causal transfer functions are stabilizable.*

Proof. Suppose that p is a causal plant in $\mathcal{P}$ expressed as $p = n/d$ with $n \in \mathcal{A}$ and $d \in \mathcal{A}\backslash\mathcal{Z}$. Let g be $\gcd(n, d)$ over $\mathbb{R}[x]$ rather than over $\mathcal{A}$. We assume without loss of generality that g is expressed as $1 + g_1 x$ with $g_1 \in \mathbb{R}$.

Let n' and d' be polynomials n/g and d/g, respectively, in $\mathbb{R}[x]$ and further n'' and d'' be polynomials in $\mathcal{A}$ defined as follows:

$$\begin{aligned}
n'' &= n'(1 + g_1 x + \tfrac{2}{9} g_1^2 x^2) \\
&= \begin{cases} n'(1 + \frac{1}{3} g_1 x)(1 + \frac{2}{3} g_1 x) & (\text{if } g_1 \neq 0), \\ n & (\text{if } g_1 = 0), \end{cases} \\
d'' &= d'(1 + g_1 x + \tfrac{2}{9} g_1^2 x^2) \\
&= \begin{cases} d'(1 + \frac{1}{3} g_1 x)(1 + \frac{2}{3} g_1 x) & (\text{if } g_1 \neq 0), \\ d & (\text{if } g_1 = 0) \end{cases}
\end{aligned}$$

(Note here that $\frac{2}{9}$ above can be other values). Then one can check that $n \in \Lambda_{pI_1}$, $d'' \in \Lambda_{pI_2}$ and the greatest common divisor of n and d'' over $\mathbb{R}[x]$ is a unit. Let α, β be in $\mathbb{R}[x]$ such that the following equation holds over $\mathbb{R}[x]$:

$$(\alpha + rd'')n + (\beta - rn)d'' = 1, \tag{3}$$

where r is an arbitrary element in $\mathbb{R}[x]$. Let n_0, d_0, α_0, and β_0 denote the constant terms of n, d'', α, and β, respectively. Similarly let α_1, β_1, α_1', and β_1' denote the coefficients of α, β, $\alpha + rd''$, and $\beta - rn$, respectively, with the degree 1 with respect to the variable x. In order to show that (3) holds over $\mathcal{A}$, we want to find r such that $\alpha_1' = \beta_1' = 0$. To do so, we let $r = (\alpha_0\beta_1 - \alpha_1\beta_0)x$. Then it is easy to check that $\alpha_1' = \beta_1' = 0$ hold from the relations that $\alpha_0 n_0 + \beta_0 d_0 = 1$ and $\alpha_1 n_0 + \beta_1 d_0 = 0$. Now that $(\alpha + rd'')$, $(\beta - rn) \in \mathcal{A}$, we have $\Lambda_{pI_1} + \Lambda_{pI_2} = \mathcal{A}$, so that the plant is stabilizable. $\square$

4.2 Construction of Stabilizing Controllers

We present here the method to construct stabilizing controllers under the cases (i) $\mathcal{A} = \mathbb{Z}[\sqrt{5}i]$ or $= \mathbb{R}[x^2, x^3]$ and (ii) single-input single-output plant. This is an application of the proof ("(iii)→(i)" part) of Theorem 2.1 in [3] (for the proof, see [2]).

Since the plant p is of the single-input single-output, the set $\mathcal{I}$ defined in Definition 2.4 of [3] is equal to $\{\{1\}, \{2\}\}$, say $I_1 = \{1\}$ and $I_2 = \{2\}$ as in the proof of Proposition 4.1. Two generalized elementary factors Λ_{pI_1} and Λ_{pI_2} are given as (2). Since in the cases $\mathcal{A} = \mathbb{Z}[\sqrt{5}i]$ and $\mathcal{A} = \mathbb{R}[x^2, x^3]$ any causal transfer functions are stabilizable, $\Lambda_{pI_1} + \Lambda_{pI_2} = \mathcal{A}$ holds by Theorem 2.1 in [3]. We should find $\lambda_{I_1} \in \Lambda_{pI_1}$ and $\lambda_{I_2} \in \Lambda_{pI_2}$ such that $\lambda_{I_1} + \lambda_{I_2} = 1$.

From Lemmas 4.7 and 4.10 of [2], there exist (right-/left-)coprime factorizations over $\mathcal{A}_{\lambda_I}$ for $I = I_1$, I_2. We let $n_1 = 1$, $d_1 = 1/p$, $n_2 = p$, $d_2 = 1$ with $p =$

$n_1/d_1 = n_2/d_2$, $n_1, d_1 \in \mathcal{A}_{\lambda_{I_1}}$, and $n_2, d_2 \in \mathcal{A}_{\lambda_{I_2}}$. Then the coprime factorizations are obtained as follows:

$$y_1 n_1 + x_1 d_1 = 1 \ \text{(over } \mathcal{A}_{\lambda_{I_1}}\text{)} \ \text{and}$$
$$y_2 n_2 + x_2 d_2 = 1 \ \text{(over } \mathcal{A}_{\lambda_{I_2}}\text{)},$$

where $y_1 = 1$, $x_1 = 0$, $y_2 = 0$ and $x_2 = 1$. When we use the parameters $r_1 \in \mathcal{A}_{\lambda_{I_1}}$ and $r_2 \in \mathcal{A}_{\lambda_{I_2}}$, we also have

$$(y_1 + r_1 d_1) \cdot n_1 + (x_1 - r_1 n_1) \cdot d_1 = 1 \ \text{(over } \mathcal{A}_{\lambda_{I_1}}\text{)},$$
$$(y_2 + r_2 d_2) \cdot n_2 + (x_2 - r_2 n_2) \cdot d_2 = 1 \ \text{(over } \mathcal{A}_{\lambda_{I_2}}\text{)}.$$

From the proof of Theorem 2.1 in [3] (Theorem 3.2 of [2]), a stabilizing controllers of the plant p is given as the following form

$$c = \frac{a_1 \lambda_{I_1}^{\omega} d_1 (y_1 + r_1 d_1) + a_2 \lambda_{I_2}^{\omega} d_2 (y_2 + r_2 d_2)}{a_1 \lambda_{I_1}^{\omega} d_1 (x_1 - r_1 n_1) + a_2 \lambda_{I_2}^{\omega} d_2 (x_2 - r_2 n_2)}, \tag{4}$$

where ω is a sufficiently large positive integer and a_1, a_2, r_1, r_2 are elements in $\mathcal{A}$ such that the following three conditions hold:

(i) The equality $a_1 \lambda_{I_1}^{\omega} + a_2 \lambda_{I_2}^{\omega} = 1$ holds.

(ii) The following matrices are over $\mathcal{A}$ for $k = 1, 2$:

$$\begin{array}{l} a_k \lambda_{I_k}^{\omega} n_k (x_k - r_k n_k),\ a_k \lambda_{I_k}^{\omega} n_k (y_k + r_k d_k), \\ \qquad a_k \lambda_{I_k}^{\omega} d_k (x_k - r_k n_k),\ a_k \lambda_{I_k}^{\omega} d_k (y_k + r_k d_k). \end{array} \tag{5}$$

(iii) The denominator of c is nonzero.

In the following we continue Examples 3.1 and 3.2 in which the stabilizing controllers are constructed.

Example 3.1 (continued) Let us stabilize the plant $p = (1 + \sqrt{5}i)/2$. We can find that $\alpha' = 3 =: \lambda_{I_1} \in \Lambda_{pI_1}$ and $\beta = 2 =: \lambda_{I_2} \in \Lambda_{pI_2}$, where α' and β are the symbols used in the proof of Proposition 4.1. By using λ_{I_1} and λ_{I_2} the stabilizing controllers are obtained in the form (4). For example, let us choose $r_1 = r_2 = 0$. Then we can select $\omega = 1$ to satisfy the condition (ii) above. The coefficients a_1 and a_2 are 1 and -1, respectively. We obtain a stabilizing controller $c = \frac{-1+\sqrt{5}i}{2}$, which is same as the stabilizing controller given in [1].

Let us present the parameterization of the stabilizing controllers according to [3]. Now $H_0 = H(p, c)$ is expressed as

$$H_0 = \begin{bmatrix} -2 & 1 + \sqrt{5}i \\ 1 - \sqrt{5}i & -2 \end{bmatrix}.$$

According to Theorem 4.3 of [3], the set of all $H(p, c)$'s with all stabilizing controllers c's, denoted by $\mathcal{H}(p; \mathcal{A})$, is given as follows:

$$\mathcal{H}(p; \mathcal{A}) = \tag{6}$$

$$\left\{ \begin{array}{l} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \middle| \; q_{11}, q_{12}, q_{21}, q_{22} \in \mathcal{A}, \\ \quad h_{11} = h_{22} = 3\sqrt{5}iq_{12} - 2\sqrt{5}iq_{21} + 6q_{11} \\ \qquad\qquad\qquad -3q_{12} - 2q_{21} + 6q_{22} - 2, \\ \quad h_{12} = -3\sqrt{5}iq_{11} + 2\sqrt{5}iq_{21} - 3\sqrt{5}iq_{22} \\ \qquad\qquad +\sqrt{5}i - 3q_{11} + 9q_{12} - 4q_{21} - 3q_{22} + 1, \\ \quad h_{21} = 2\sqrt{5}iq_{11} - 2\sqrt{5}iq_{12} + 2\sqrt{5}iq_{22} - \sqrt{5}i \\ \qquad\qquad\qquad -2q_{11} - 4q_{12} + 4q_{21} - 2q_{22} + 1, \\ \quad h_{11} \text{ and } h_{22} \text{ are nonzero.} \end{array} \right\}.$$

In the set $\mathcal{H}(p; \mathcal{A})$ in (6), q_{ij} is corresponding to the (i, j)-entry of the matrix Q used in § 4 of [3]. One can observe that $(1 + \sqrt{5}i)h_{11} = -2h_{12}$, so that $p = -h_{12}h_{11}^{-1}$. By Corollary 4.1 of [3] any stabilizing controllers are expressed as $h_{11}^{-1}h_{21}$ or equivalently $h_{21}h_{22}^{-1}$ provided that h_{11} and h_{22} are nonzero.

Example 3.2 (continued) Let us construct stabilizing controllers of the plant $p = (1 - x^3)/(1 - x^2)$. In this case, g, n and d'' used in the proof of Proposition 4.2 are $1 - x$, $1 - x^3$ and $1 - \frac{7}{9}x^2 + \frac{2}{9}x^3$, respectively. We calculate α and β used in the proof satisfying (3). They are computed as $\alpha = -\frac{101}{988} - \frac{441}{988}x + \frac{77}{494}x^2$ and $\beta = \frac{1089}{988} + \frac{441}{988}x + \frac{693}{988}x^2$. By letting $r = \frac{441}{988}x$, all of $(\alpha + rd'')$, $(\beta - rn)$, n, d'' in (3) become in $\mathcal{A}$.

Let $\lambda_{I_1} = n \in \Lambda_{pI_1}$ and $\lambda_{I_2} = d'' \in \Lambda_{pI_2}$. Then a stabilizing controllers is given in the form of (4). For example, let us choose $r_1 = r_2 = 0$. Then we can select $\omega = 1$ to satisfy the condition (ii). The coefficients a_1 and a_2 are $-\frac{101}{988} + \frac{77}{494}x^2 - \frac{343}{988}x^3 + \frac{49}{494}x^4$, and $\frac{1089}{988} + \frac{693}{988}x^2 + \frac{441}{988}x^4$, respectively. Finally, from the formula of (4) we obtain a stabilizing controller given as follows:

$$c = \frac{-101 + 255x^2 - 343x^3 - 56x^4 + 343x^5 - 98x^6}{1089 - 154x^2 + 242x^3 - 98x^4 + 154x^5 - 343x^6 + 98x^7}.$$

Again, let us consider the parameterization of the stabilizing controllers. Although we can state the set $\mathcal{H}(p; \mathcal{A})$, we do not state it unlike (6) because of space limitations. Further any stabilizing controllers are expressed as $h_{11}^{-1}h_{21}$ or equivalently $h_{21}h_{22}^{-1}$ provided that h_{11} and h_{22} are nonzero.

5 Concluding Remarks

In this paper, we have presented a generalization of Anantharam's result and given a condition of the nonexistence of the right-/left-coprime factorizations of stabilizable plants. As examples satisfying the obtained condition, two models were presented. We have also presented a method to construct stabilizing controllers of stabilizable single-input single-output plants of such models.

Propositions 3.1 and 3.3 give the conditions of the existence/nonexistence of the doubly coprime factorizations of stabilizable plants. However they do not characterize the commutative ring $\mathcal{A}$ on which there exist the doubly coprime factorizations of stabilizable plants. This problem is not solved yet.

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